$$\mathrm{Var}(X+Y) = \mathrm{Var}\,X + \mathrm{Var}\,Y + 2\,\mathrm{cov}(X,Y)$$

Un'urna contiene a palline rosse e b palline nere. Si eseguono 2 estrazioni senza r. Sia

U = n° di palline rosse estratte nelle 2 estr.

Calcolare $E[U]$ e $\mathrm{Var}\,U$.

$U = X + Y$, dove

$$X = \begin{cases} 1 & \text{se alla } 1^a \text{ estr. esce p. rossa} \\ 0 & \text{se no} \end{cases}$$

$$Y = \begin{cases} 1 & \text{se alla } 2^a \text{ estr. esce p. rossa} \\ 0 & \text{se no} \end{cases}$$

$$U = X + Y$$

$$E[U] = E[X] + E[Y] = \frac{a}{a+b} \cdot 2$$

$$X \sim B\left(1, \frac{a}{a+b}\right) \quad ; \quad Y \sim B\left(1, \frac{a}{a+b}\right)$$

$$\operatorname{Var} \bar{U} = \operatorname{Var}(X+Y) = \underline{\operatorname{Var} X} + \underline{\operatorname{Var} Y} + 2\operatorname{cov}(X,Y)$$

$$P(A \cap B) = P(B|A)P(A)$$

$$\underline{p(1,1)} = P(X=1, Y=1) =$$

$$= P(Y=1 \mid X=1)\, P(X=1) = \frac{a-1}{a+b-1} \cdot \frac{a}{a+b} \neq$$

$$\neq P(X=1) \cdot P(Y=1) = \left(\frac{a}{a+b}\right)^2$$

$$= p_X(1) \cdot p_Y(1)$$

$$\mathrm{Var}\, X = \frac{a}{a+b}\left(1-\frac{a}{a+b}\right) = \frac{ab}{(a+b)^2} \qquad X \sim B\left(1, \frac{a}{a+b}\right)$$

$$\mathrm{Var}\, Y = \frac{ab}{(a+b)^2} \qquad Y \sim B\left(1, \frac{a}{a+b}\right)$$

$$\longrightarrow \mathrm{Cov}(X,Y) = \underbrace{E[XY]}_{\downarrow} - E[X]E[Y] =$$

$$= \bullet - \left(\frac{a}{a+b}\right)^2$$

$$XY = \begin{cases} 1 & \text{se esce p. rosso in entrambe le estr.} \\ 0 & \text{se no} \end{cases}$$

$$\mathrm{Var}\, U = \mathrm{Var}\, X + \mathrm{Var}\, Y + 2\,\mathrm{cov}(X, Y) =$$

$$= \mathrm{Var}\, X + \mathrm{Var}\, Y + 2\left(E[XY] - \overline{E[X]E[Y]}\right) =$$

$$= \frac{2ab}{(a+b)^2} + 2\left(\frac{a(a-1)}{(a+b)(a+b-1)} - \left(\frac{a}{a+b}\right)^2\right) =$$

$$\ldots\ldots$$

Funzione di ripartizione.

Sia X una v.a definita su $(\Omega, \mathcal{A}, P)$

Definizione Si chiama funzione di ripartizione (o di distribuzione, distribution function) la funzione $F: \mathbb{R} \longrightarrow \mathbb{R}$ definita da

$$F(t) = P(X \leq t), \quad t \in \mathbb{R}$$

$$\{X \leq t\} \qquad \{X \in I\}$$

Esempio. Sia $X \sim B(2, p)$ | $X\ Y\ Z$

F_X, F_Y

F_Z

$F(t) = P(X \leq t)$

$$P(X = k) = \begin{cases} \binom{2}{k} p^k (1-p)^{2-k} & k = 0, 1, 2 \\ 0 & \cancel{k \neq 0} \end{cases}$$

$$X = \begin{cases} 0 & (1-p)^2 \\ 1 & 2p(1-p) \\ 2 & p^2 \end{cases}$$

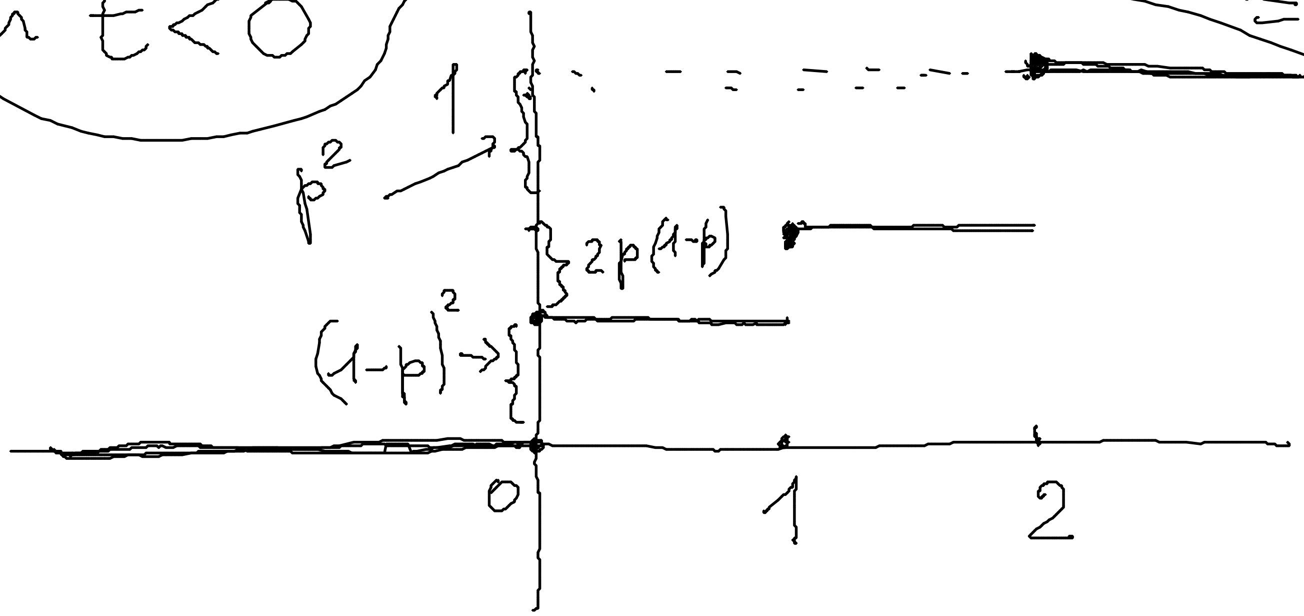
$$P(X \le 1.2) = \sum_{x \le 1.2} p_X(x) =$$

$$= P(X=0) + P(X=1) =$$

$$= (1-p)^2 + 2p(1-p)$$

$$P(X \le t) = 0$$

for $t<0$



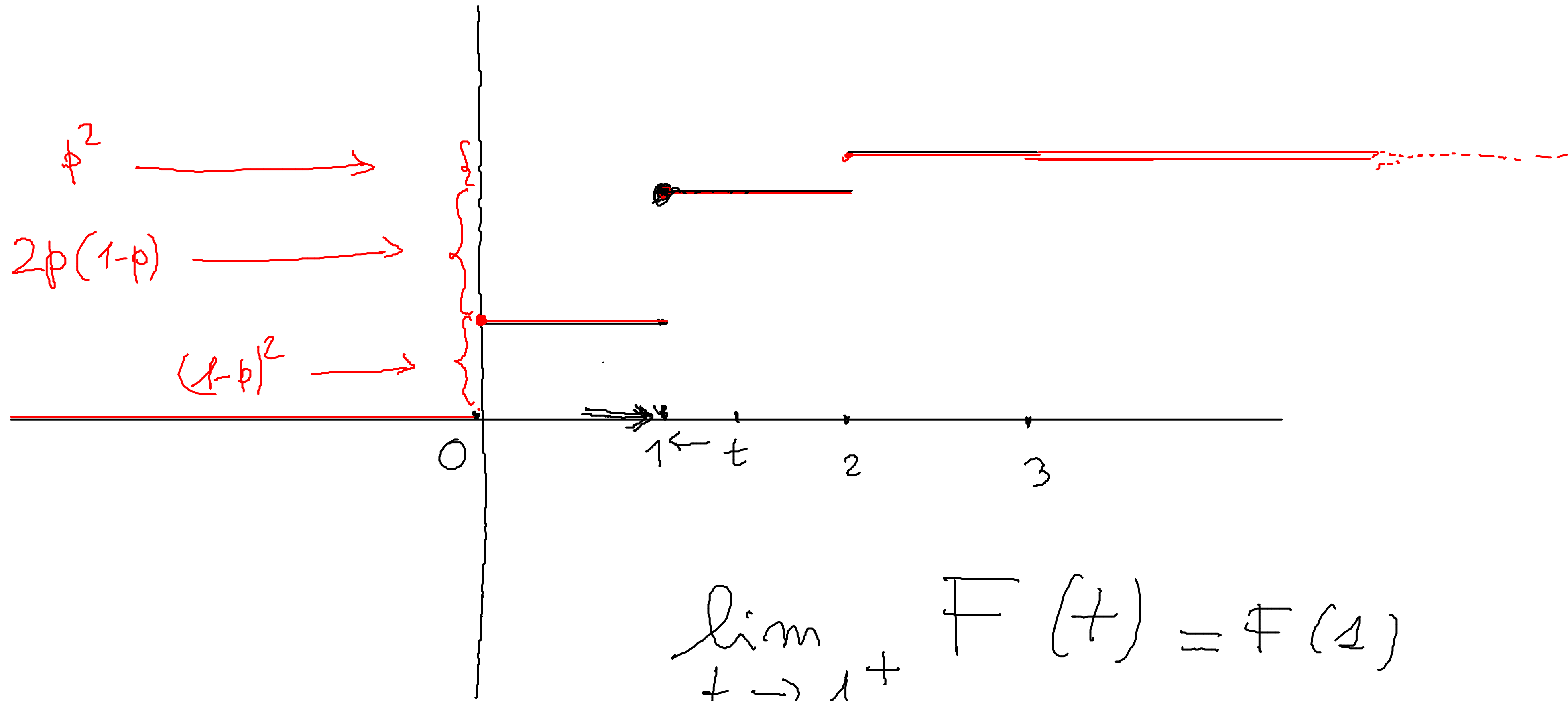
$t = 0$

$P(X \le 0)$

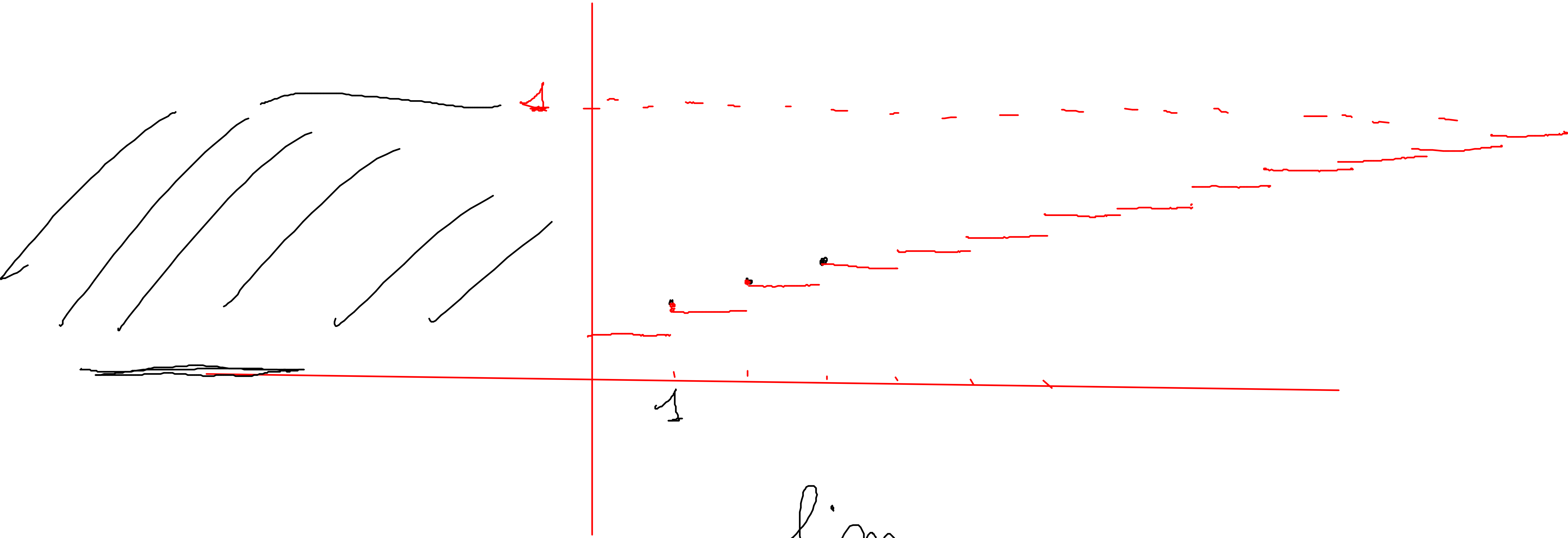
$P(X \le 0.3)$

$P(X \le 1) =$

$$P(X \le 2) = (1-p)^2 + 2p(1-p) + p^2 = 1$$



$$\lim_{t \to 1^+} F(t) = F(1)$$



$\lim_{t \to}$

Proposizione. Sia X una v. a. e F la sua f. d. r. Allora F ha le seguenti proprietà

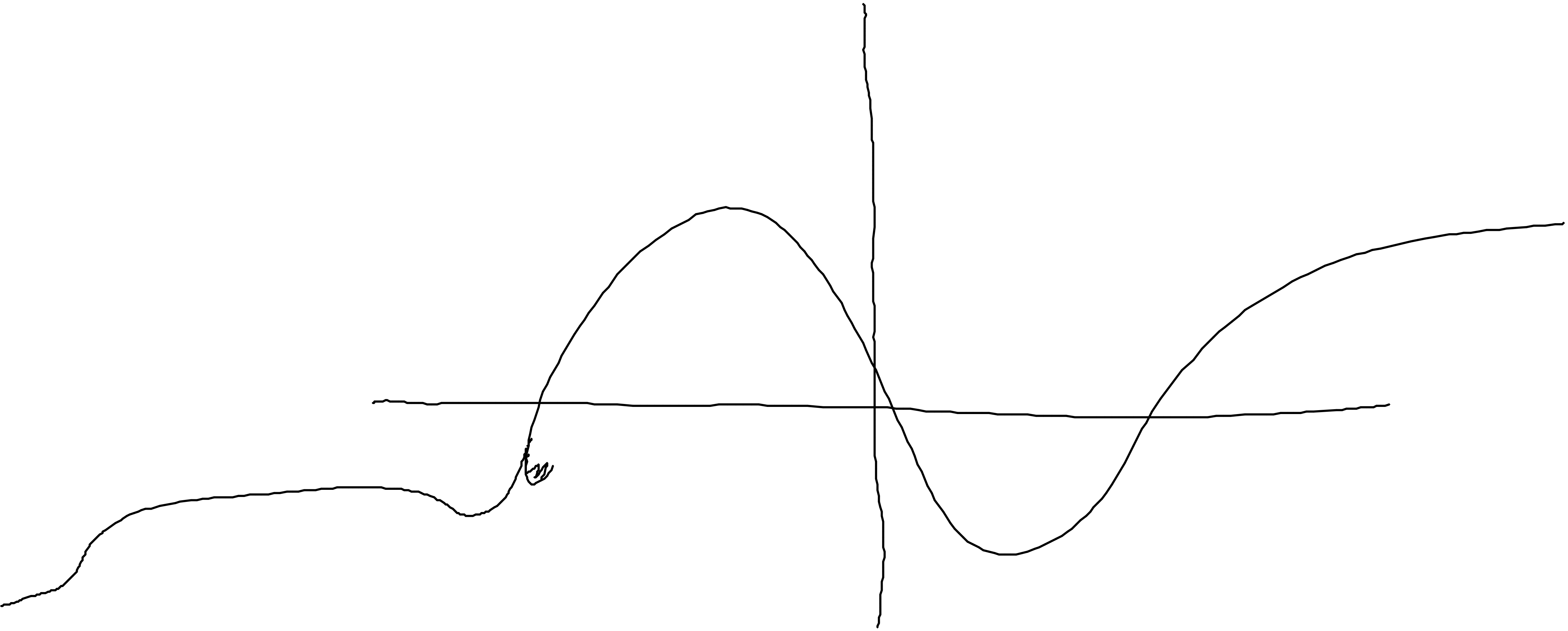
$F(t) = P(X \le t)$

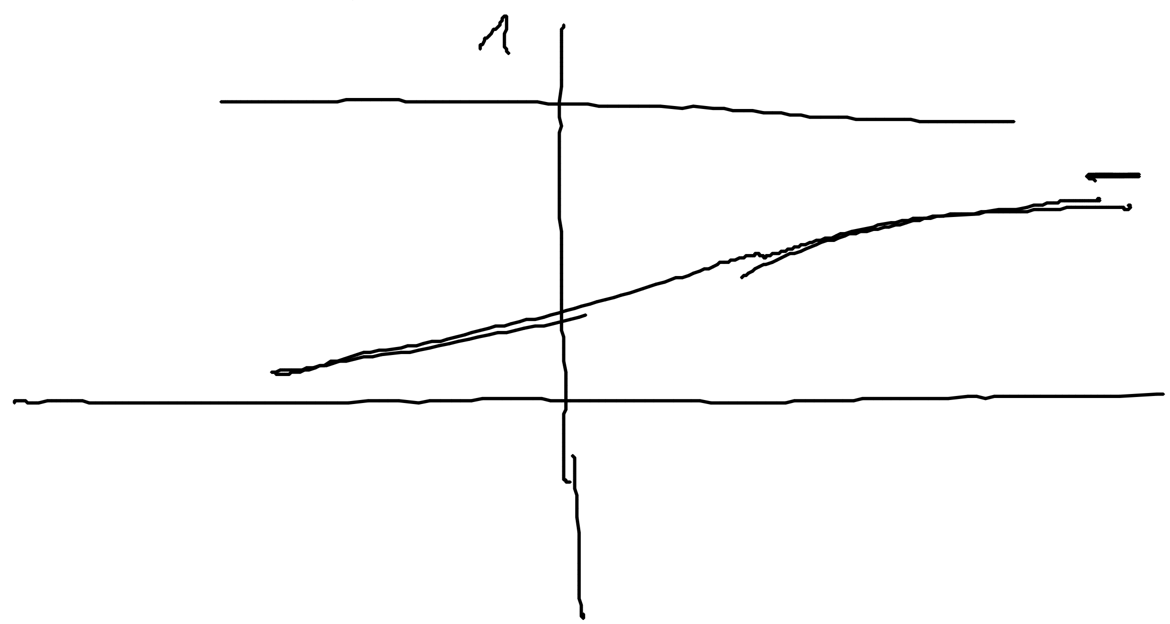
(i) $0 \le F(t) \le 1 \quad \forall t$

(ii) F è non decrescente
$(x \le y \Rightarrow F(x) \le F(y))$

(iii) $\lim_{t \to +\infty} F(t) = 1$; $\lim_{t \to -\infty} F(t) = 0$

(iv) F è continua a destra: $\forall x \in \mathbb{R}$
$F(x) = \lim_{t \to x^+} F(t)$





Calc. $P(X \in I)$ $\quad I$ intervallo

conoscendo $\quad F(t) = P(X \leq t) \quad \forall t$

1) $\quad I = (a, b]$

$$P(X \in I) = \underbrace{P(a < X \leq b)}_{} =$$

$$= P(X \leq b) - P(X \leq a) =$$

$$= F(b) - F(a)$$

2) $P(X > a) = 1 - P(X \leq a)$

$= 1 - F(a)$

3) $P(X < a) = \lim_{t \to a^-} F(t)$

4) $P(a \leq X \leq b) =$

$= P(X \leq b) - P(X < a)$

$= \underline{F(b)} - \underline{\lim_{t \to a^-} F(t)}$

5) $$P(a < X < b) = P(X < b) - P(X \leq a) = \lim_{t \to b^-} F(t) - F(a)$$

6) $$P(a \leq X < b) = \lim_{t \to b^-} F(t) - \lim_{t \to a^-} F(t)$$

7) $$P(X \geq a) = 1 - \lim_{t \to a^-} F(t)$$

8) $P(X=a) =$

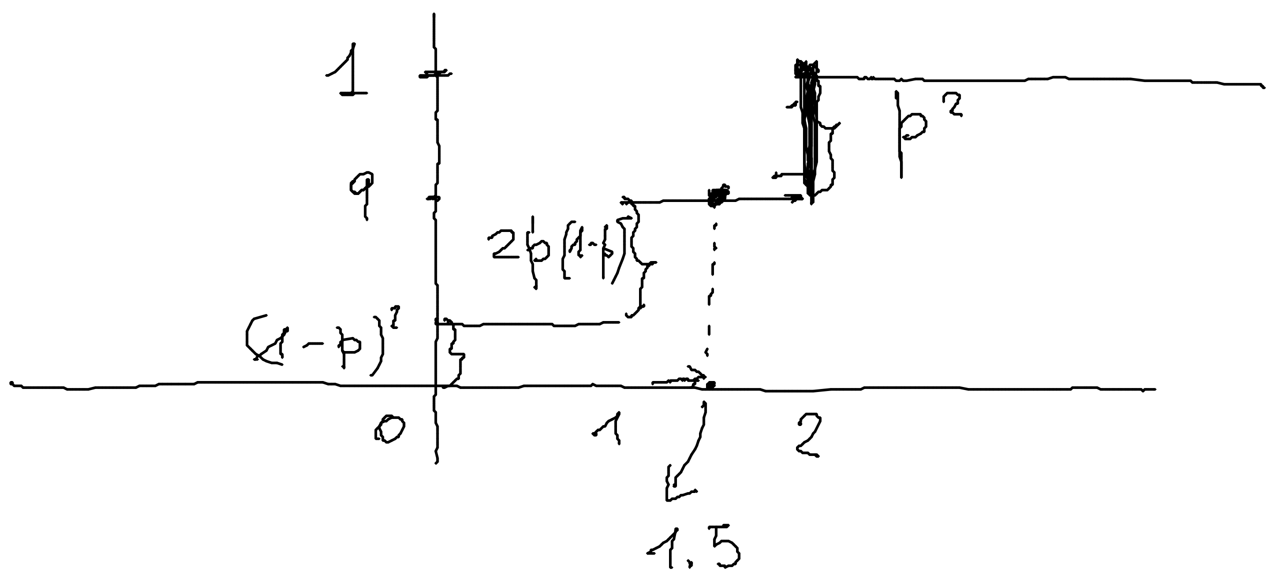
$= P(a \le X \le a) -$

$= \boxed{F(a) - \lim\limits_{t \to a^-} F(t) = P(X=a)}$

$P(X=1.5) =$

$= F(1.5) - \lim\limits_{t \to 1.5} F(t)$

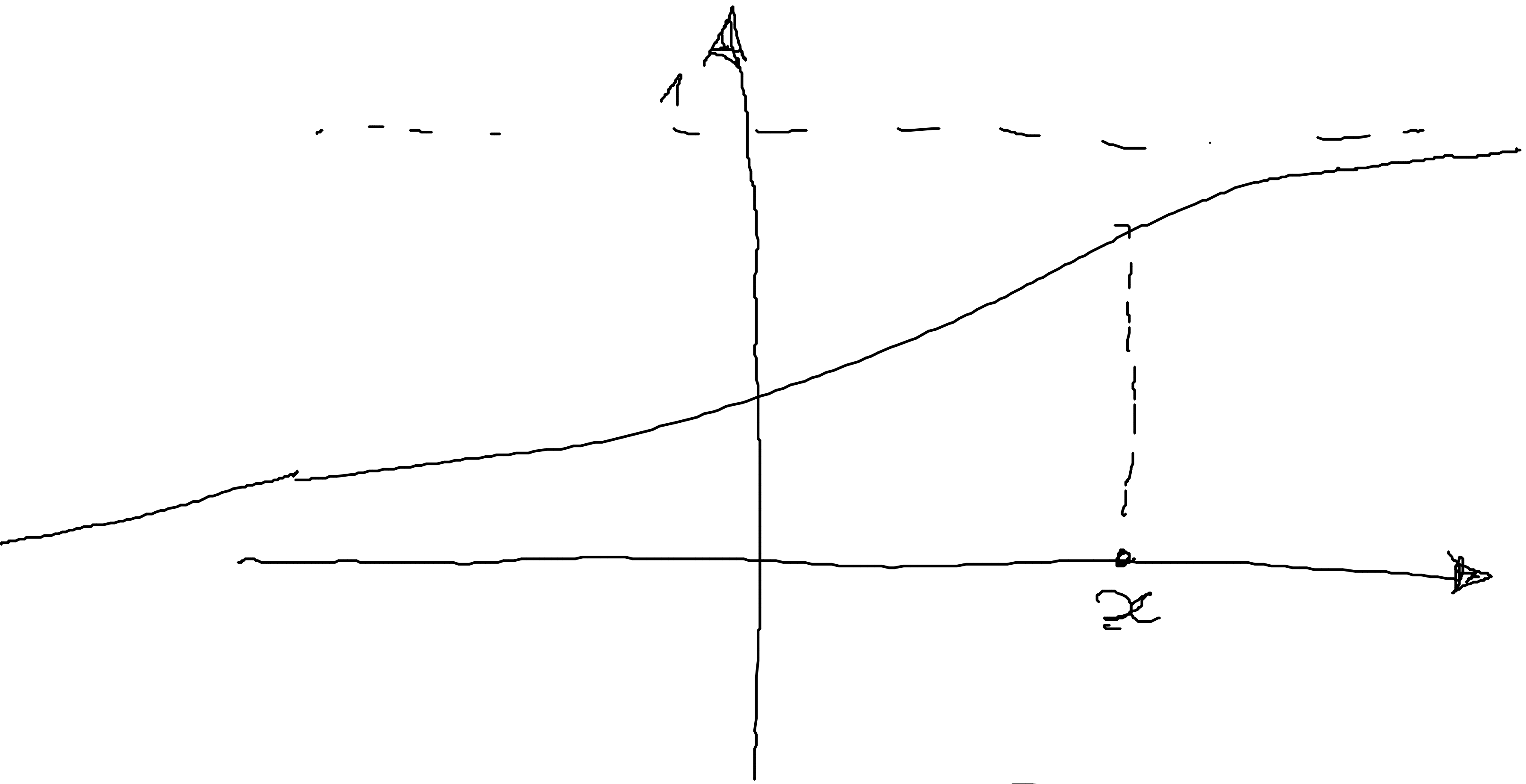
$= q - q = 0$



$P(X=2) =$

$= 1 - (1-p)^2 - 2p(1-p)$

$= p^2$



$$\forall x \in \mathbb{R}$$

$$P(X = x) = 0$$

Definizione. Si dice che X è una v. a. continua se è continua la sua f. d. r. F o, in modo equivalente, se $\forall x \in R$, $P(X=x)=0$

$$P(X=x) = F(x) - \lim_{t \to x^-} F(t) =$$

$$= \lim_{t \to x^+} F(t) - \lim_{t \to x^-} F(t)$$

Variabili aleatorie assolutamente continue

X discreta $\to$ $\boxed{P(X \in A) = \sum_{x \in A} p(x)}$

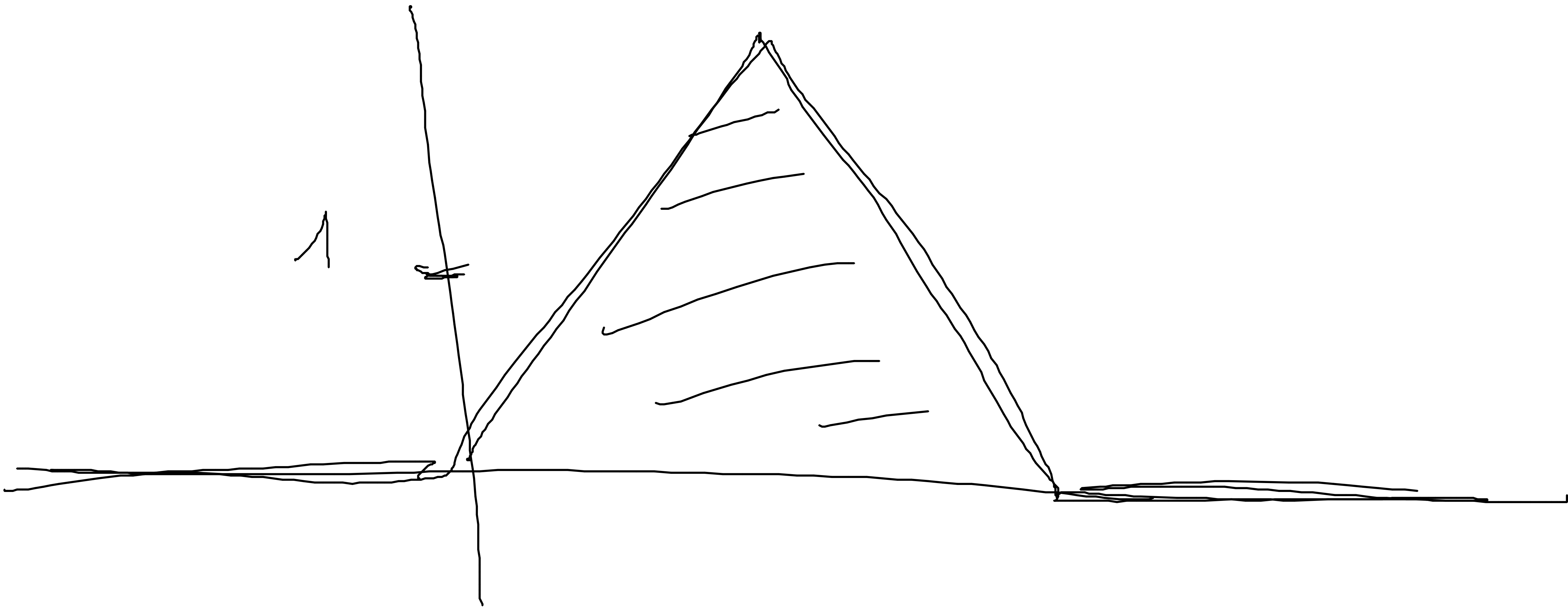
Definizione. La v.a. X si dice assolutamente continua se esiste una funzione $f : R \to R^+$, integrabile su $\mathbb{R}$ tale che, $\forall A$ intervallo (misurabile), si abbia

$$\boxed{P(X \in A) = \int_A f(x)\,dx}$$

Osservazione. f (densità di X v. a. assolutamente continua) non è unica

$$P(X \in A) = \int_A f(x)\,dx$$

$$0 \leq p(x) \leq 1 \qquad f$$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X ha media finita

Caso discreto

$$\sum_{x \in \mathbb{R}} |x| \, p(x) < \infty$$

Caso ass. cont

$$\int_{\mathbb{R}} |x| \, f(x) \, dx < +\infty$$

$$E[X] = \sum_{x \in \mathbb{R}} x \, p(x)$$

$$E[X] = \int_{\mathbb{R}} x \, f(x) \, dx$$

$$E[X^k] = \sum_{x \in \mathbb{R}} x^k \, p(x)$$

$$E[X^k] = \int_{\mathbb{R}} x^k \, f(x) \, dx$$

$$E[\varphi(X)] = \sum_{x \in \mathbb{R}} \varphi(x) p(x) \quad \Big| \quad E[\varphi(X)] = \int_{\mathbb{R}} \varphi(x) f(x) dx$$

$$\text{Var } X = E[(X - E[X])^2] = E[X^2] - E^2[X]$$

$$= \sum_{x \in \mathbb{R}} (x - E[X])^2 p(x) \quad \Big| \quad \int_{\mathbb{R}} (x - E[X])^2 f(x) dx$$

$$= \sum x^2 p(x) - \left(\sum x p(x) \right)^2 \quad \Big| \quad \int x^2 f(x) dx - \left(\int x f(x) dx \right)^2$$

$$>$$

Caso discreto

$$P(X \in A) = \sum_{x \in A} p(x)$$

Caso assolutamente continuo

$$P(X \in A) = \int_A f(x)\,dx$$

$$A = (3,8)$$

$$\int_A = \int_3^8$$

f si chiama "densità" di probabilità di X.

$$\int_{\mathbb{R}} f(x)\, dx = P(X \in \mathbb{R}) = P(\Omega) = 1$$

$$\sum_{x \in \mathbb{R}} p(x) = 1 \qquad p(x) = P(X = x)$$

Osservazione. Se X è assolutamente continua, allora X è continua.

Il viceverso è falso: esistono v. a. continue che non sono assolutamente continue.

Dimostraz.

$$F(t) = P(X \leq t) = P(X \in (-\infty, t]) = \int_{-\infty}^{t} f(x)\,dx$$

Dim. di (ii)

$$x < y \Rightarrow F(x) \leq F(y)$$

$$F(x) = P(X \leq x)$$

$$F(y) = P(X \leq y)$$

$$A \subseteq B$$

$$\{X \leq x\} \subseteq \{X \leq y\}$$

$$\omega : X(\omega) \leq x < y$$

$$F(x) = P(X \leq x) \leq P(X \leq y) = F(y)$$

$p(x) = P(X = x)$

Nel caso assolut. continuo

$P(X = x) = 0$ perché

le v. a. ass. cont. sono continue

$f(x)$

$$P(\text{esce rosso in entrambe le estraz}) =$$

$$= P(X=1, Y=1) = \frac{a(a-1)}{(a+b)(a+b-1)}$$

$$XY = \begin{cases} 1 & \text{con prob } \dfrac{a(a-1)}{(a+b)(a+b-1)} \\ & \\ 0 & \text{con prob } 1 - \dfrac{\cdot\ \text{-}\ \text{-}}{\cdot\ \text{—}} \end{cases}$$

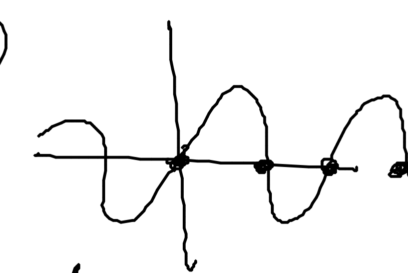
$$E[XY] = \frac{a(a-1)}{(a+b)(a+b-1)}$$

Dim di (iii).

$$\lim_{t\to+\infty} P(X \le t) = 1$$

$$\lim_{n\to\infty} P(X \le n) = 1$$

$$\lim_{t\to\infty} F(t) \qquad \lim_{n\to\infty} F(n)$$



$$F(t) = \operatorname{sen} \pi t \qquad \text{non esiste}$$

$$F(n) = 0 \qquad \lim_{t\to\infty} F(t)$$

$$\lim_{n\to\infty} P(\{X \leq n\}) =$$

$$A_n = \{X \leq n\}$$

$$A_n \subseteq A_{n+1}$$ è una succ. crescente di eventi

$$\lim_{n\to\infty} P(A_n) = P\Big(\underbrace{\bigcup_n A_n}\Big) = P(\Omega) = 1$$

$$\bigcup_n \{X \leq n\} = \Omega$$